Closing Fri: 2.7, 2.7-8
Closing Tues: 2.8
Closing next Thurs: 3.1-2
Closing next Fri: 3.3 (last before Exam1)

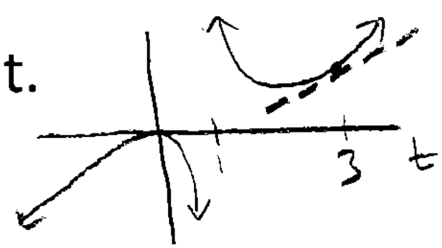
2.7-8 Derivatives

Recall: We defined the slope of the tangent line to $f(x)$ at $x = a$ by

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

We call this value the **derivative** and denote it by $f'(a)$.

(From HW 2.7-8/6)
Entry Task: An object is moving on a straight line and its position is given by $p(t) = \frac{t^2}{t-1}$ feet.



Find $p'(3)$.

NOTE: IN HW REPLACE "3" WITH "a"

$$\lim_{h\to 0}\frac{p(3+h)-p(3)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{(3+h)^2}{3+h-1}-\frac{3^2}{3-1}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{(3+h)^2}{2+h}-\frac{9}{2}}{h} \qquad \frac{2(2+h)}{2(2+h)}$$

$$= \lim_{h\to 0}\frac{2(3+h)^2-9(2+h)}{2h(2+h)}$$

$$= \lim_{h\to 0}\frac{2(9+6h+h^2)-18-9h}{2h(2+h)}$$

$$= \lim_{h\to 0}\frac{\cancel{18}+12h+2h^2-\cancel{18}-9h}{2h(2+h)}$$

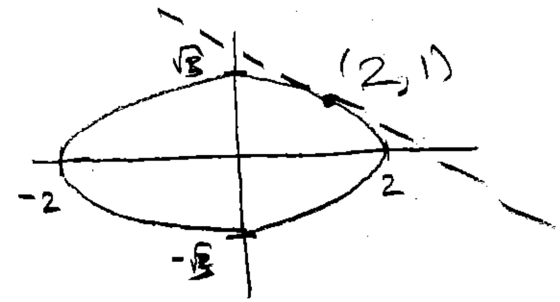
$$= \lim_{h\to 0}\frac{3h+2h^2}{2h(2+h)}$$

$$= \lim_{h\to 0}\frac{\cancel{h}(3+2h)}{2\cancel{h}(2+h)}$$

$$= \frac{3+0}{2(2+0)} = \boxed{3/4} \quad \text{ft/sec}$$

slope

(Like HW 2.7/6)



Example: Consider the ellipse

$$x^2 + 2y^2 = 6$$

Find the slope of the tangent line at

$$(x, y) = (2, 1)$$

$$2y^2 = 6 - x^2$$

$$y^2 = \frac{1}{2}(6 - x^2) = 3 - \frac{1}{2}x^2$$

$$y = \pm\sqrt{3 - \frac{1}{2}x^2} = f(x)$$

WANT

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$\lim_{h \to 0} \frac{\sqrt{3 - \frac{1}{2}(2+h)^2} - \sqrt{3 - \frac{1}{2}(2)^2}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{3 - \frac{1}{2}(2+h)^2} - 1}{h} \cdot \frac{\sqrt{3 - \frac{1}{2}(2+h)^2} + 1}{\sqrt{3 - \frac{1}{2}(2+h)^2} + 1}$$

$$= \lim_{h \to 0} \frac{3 - \frac{1}{2}(2+h)^2 - 1}{h\left(\sqrt{3 - \frac{1}{2}(2+h)^2} + 1\right)}$$

$$= \lim_{h \to 0} \frac{3 - \frac{1}{2}(4 + 4h + h^2) - 1}{h\left(\sqrt{3 - \frac{1}{2}(2+h)^2} + 1\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{3} \cancel{- 2} - 2h - \frac{1}{2}h^2 \cancel{- 1}}{\cancel{h}\left(\sqrt{3 - \frac{1}{2}(2+h)^2} + 1\right)}$$

$$= \frac{-2 - 0}{\sqrt{3 - 2} + 1} = \frac{-2}{2} = \boxed{-1}$$

2.8 The derivative function

Example: Let $f(x) = 2x^2 - 3x$

1. Find $f'(3)$.
2. Find $f'(x)$.

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h\to 0} \frac{(2(3+h)^2 - 3(3+h)) - \overbrace{(2(3)^2 - 3(3))}^{18 \quad -9}}{h}$$

$$= \lim_{h\to 0} \frac{2(9+6h+h^2) - 9 - 3h - 9}{h}$$

$$= \lim_{h\to 0} \frac{1\cancel{8} + 12h + 2h^2 - \cancel{9} - 3h - \cancel{9}}{h}$$

$$= \lim_{h\to 0} \frac{9\cancel{h} + 2h^{\cancel{2}}}{\cancel{h}}$$

$$= 9 + 0 = \boxed{9}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(2(x+h)^2 - 3(x+h)) - (2x^2 - 3x)}{h}$$

$$= \lim_{h\to 0} \frac{2(x^2+2xh+h^2) - 3x - 3h - 2x^2 + 3x}{h}$$

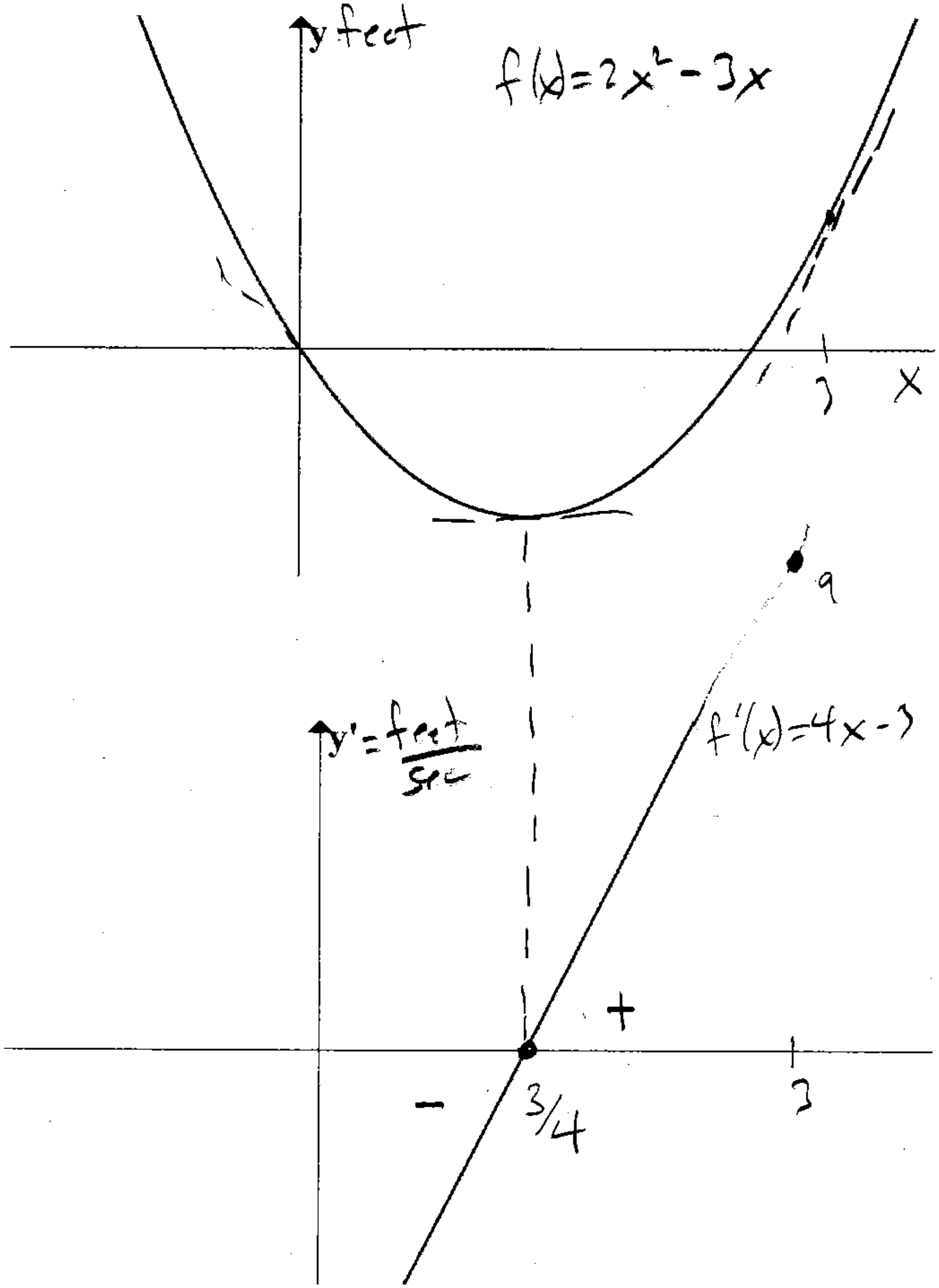
$$= \lim_{h\to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{3x} - 3h - \cancel{2x^2} + \cancel{3x}}{h}$$

$$= \lim_{h\to 0} 4x + 2h - 3$$

$$= \boxed{4x - 3}$$

NOTE: PLUG IN $x = 3$ AND YOU GET

$4(3) - 3 = \boxed{9}$ ✓



Notes/Observations:

Given $y = f(x)$.

- $y = f'(x)$ is a new function.
- $f(x)$ = "height of the graph at *x*"
- $f'(x)$ = "slope of $f(x)$ at x"
- Again, $f'(x)$ is the "instantaneous rate of change" (speedometer speed)
- The units of $f'(x)$ are $\frac{\text{y-units}}{\text{x-units}}$.

Fundamental to all applications:

$y = f(x)$	$y = f'(x)$
horiz. tangent	zero
increasing	positive
decreasing	negative

Example:

$$g(x) = \frac{2x}{x+3}$$

1. Find $g'(2)$.
2. Find $g'(x)$.

$$g'(2) = \lim_{h\to 0} \frac{g(2+h) - g(2)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{2(2+h)}{2+h+3} - \frac{2(2)}{2+3}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{4+2h}{5+h} - \frac{4}{5}}{h} \qquad \frac{5(5+h)}{5(5+h)}$$

$$= \lim_{h\to 0} \frac{5(4+2h) - 4(5+h)}{5h(5+h)}$$

$$= \lim_{h\to 0} \frac{20 + 10h - 20 - 4h}{5h(5+h)}$$

$$= \lim_{h\to 0} \frac{6}{5(5+h)} = \boxed{\frac{6}{25}}$$

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{2(x+h)}{x+h+3} - \frac{2x}{x+3}}{h} \qquad \frac{(x+h+3)(x+3)}{(x+h+3)(x+3)}$$

$$= \lim_{h\to 0} \frac{(2x+2h)(x+3) - 2x(x+h+3)}{h(x+h+3)(x+3)}$$

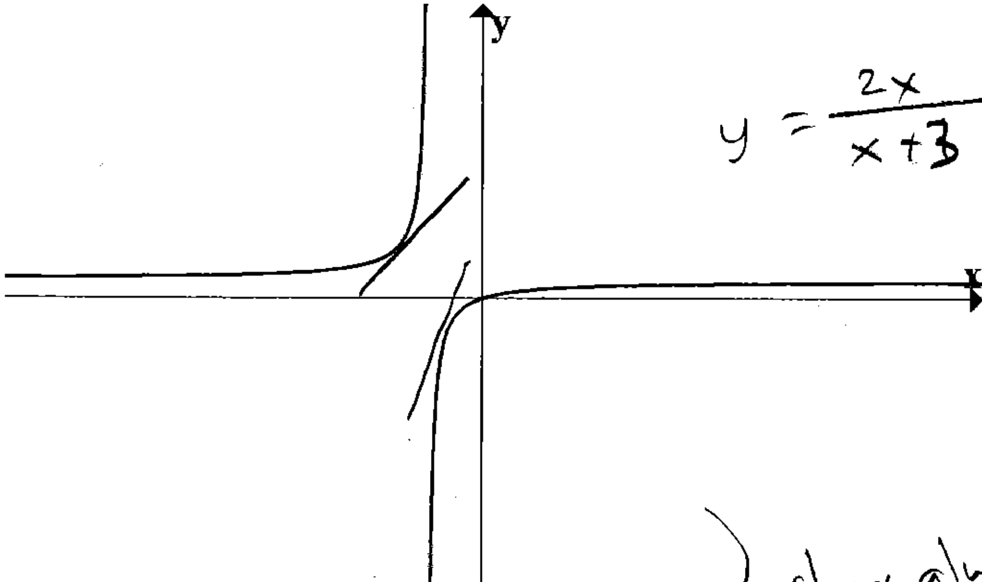
$$= \lim_{h\to 0} \frac{\cancel{2x^2} + \cancel{6x} + \cancel{2xh} + 6h - \cancel{2x^2} - \cancel{2xh} - \cancel{6x}}{h(x+h+3)(x+3)}$$

$$= \lim_{h\to 0} \frac{6\cancel{h}}{\cancel{h}(x+h+3)(x+3)}$$

$$= \lim_{h\to 0} \frac{6}{(x+h+3)(x+3)}$$

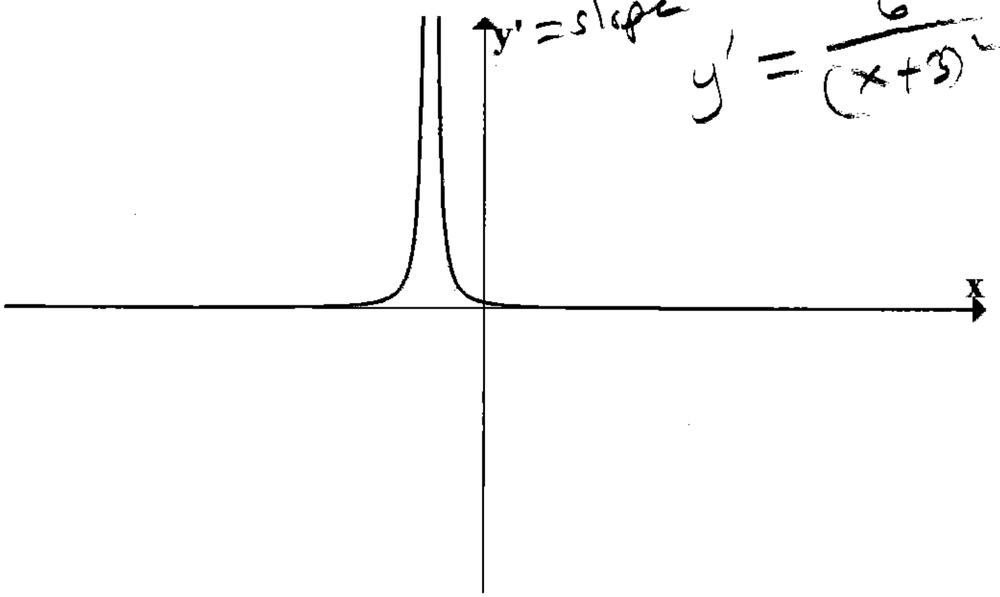
$$= \frac{6}{(x+3)(x+3)} = \boxed{\frac{6}{(x+3)^2}}$$

NOTE: $x = 2$ $\qquad \frac{6}{25}$



$$y = \frac{2x}{x+3}$$

slopes always positive



$$y' = \frac{6}{(x+3)^2}$$

Notation:

Early we found

if $\quad f(x) = 2x^2 - 3x,$

then $\quad f'(x) = 4x - 3.$

Other ways to write this include:

$$y' = 4x - 3$$

$$\frac{dy}{dx} = 4x - 3$$

$$\frac{d}{dx}(2x^2 - 3x) = 4x - 3.$$

Later we will also discuss:

$$f''(x) = y'' = \frac{d(dy/dx)}{dx} = \frac{d^2y}{dx^2}$$

Example:

if $\quad y = f(x) = 2x^2 - 3x,$

then $\quad y' = f'(x) = 4x - 3$

and $\quad y'' = f''(x) = 4$

which can also be written as

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(4x - 3) = 4$$

Differentiability

Sometimes we can have a place where "slope of tangent" doesn't make sense.

Definition:

We say a function, $y = f(x)$ is **differentiable** at $x = a$ if the following limit exists:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

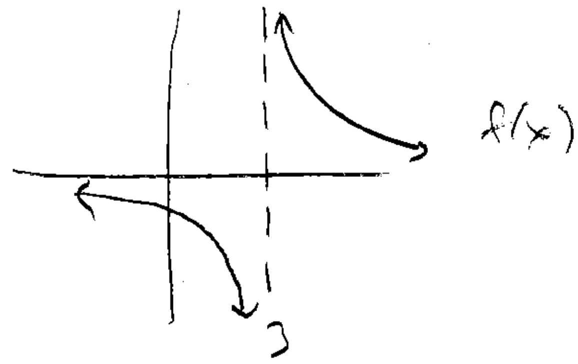
Otherwise it is not differentiable at $x = a$.

In order to get differentiable:

1. It must be defined at x = a.
2. It must be continuous at x = a.
3. The "slope" must be the same from both sides.

Examples:

1. $f(x) = \frac{1}{x-3}$

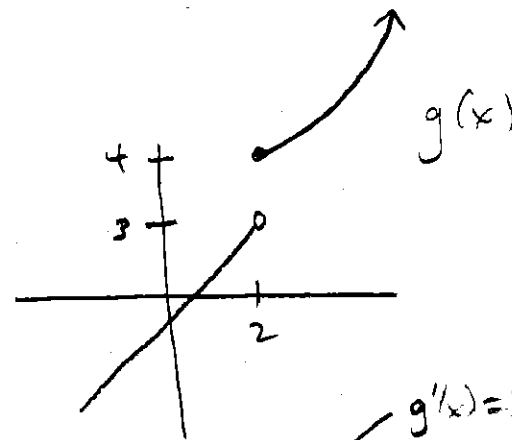


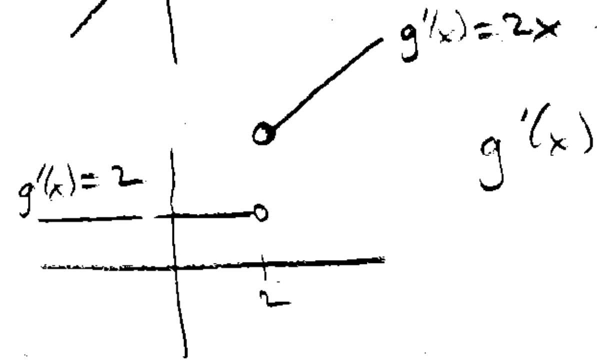
$f(x)$ is not defined at $x = 3$.

Thus, it is not continuous at $x = 3$.

And it is not differentiable at $x = 3$.

2. $g(x) = \begin{cases} 2x-1 & \text{, if } x < 2; \\ x^2 & \text{, if } x \geq 2. \end{cases}$





$g(x)$ is defined at $x = 2$.($g(2) = 4$)

But, g(x) is not cont. at $x = 2$

because $\lim_{x\to 2^-} g(x) = 3$ and

$\lim_{x\to 2^+} g(x) = 4$.

Thus, g(x) is not differentiable at x=2.

3. $k(x) = |x|$

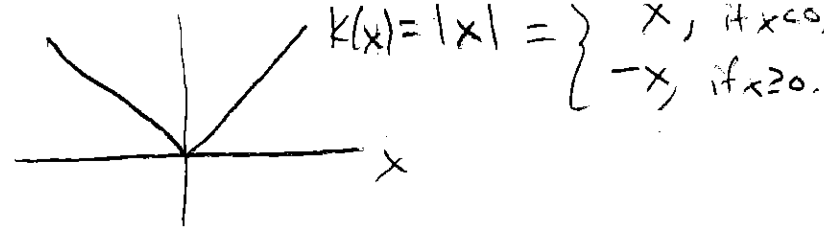
$k(x)$ is defined at $x = 0$. (k(0) = 0)

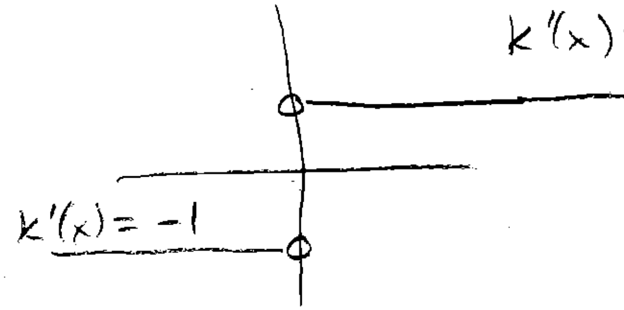
$k(x)$ is continuous at $x = 0$.

But k(x) is not differentiable at x = 0.

(The slope from the left is -1 and the slope from the right is +1)

There is a “sharp point” at x = 0.





4. $j(x) = x^{1/3}$

$j(x)$ is defined at $x = 0$. (*j*(0) = 0)

$j(x)$ is continuous at $x = 0$.

But j(x) is not differentiable at x = 0.

(The slope goes to infinity as you get close to 0).

There is a vertical tangent at x=0.